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A Finite Algorithm for a Class of Nonlinear Multiplicative Programs

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Abstract. The nonconvex problem of minimizing the product of a strictly convex quadratic function and the p-th power of a linear function over a convex polyhedron is considered. Some theoretical properties of the problem, such as the existence of minimum points and the generalized convexity of the objective function, are deepened on and a finite algorithm which solves the problem is proposed.

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1. Introduction

In this paper we consider nonlinear multiplicative problems having a polyhedral feasible region and with the objective function given by the product of a strictly convex quadratic function and the $p-$th power of a linear one, where $p \in \mathbb{R}$, $p \neq 0$. Note that the considered problems are convex multiplicative ones for $p \notin [0, 1]$, while for $p \in]0, 1]$ the objective function is the product of a strictly convex function and a concave one. Note also that this class of programs covers both quadratic multiplicative problems (case $p > 0$) and quadratic fractional ones (case $p < 0$).

The problems considered in this paper have been useful for applicative problems. For example, it is known that quadratic fractional programs have been used in application models,* such as risk theory, portfolio selection and location models (see for example [1, 2, 17, 25], see also [20] for other references on applications of fractional programs).

For this reason, quadratic fractional and generalized fractional problems have been studied in the literature from both a theoretical and an algorithmic point of view (see for all [1, 2, 3, 4, 5, 9, 11, 12, 18, 19, 20, 21]) and many results and

* Usually, the quadratic function represents the uncertainty by means of the use of a covariance matrix, while the linear function provides the expected revenues or costs.

solution algorithms appeared recently for multiplicative programs (see for example [12, 13, 14, 15, 16, 22, 23, 24]).

The aim of this paper is to propose a unifying approach to solve the considered class of quadratic multiplicative programs for all the possible values of the parameter $p \in \mathbb{R}$, $p \neq 0$. This will be done by means of the so called “optimal level solutions” method, used in the literature to solve various classes of problems (see for all [4, 6, 7, 8, 10, 12, 22]). It is worth recalling that this parametrical approach determines an optimal solution (or the infimum value) by means of simplex-like computations and without the use of any branch-and-bound iterations.

In Section 2 we define the problem and we prove that the objective function is strictly pseudoconvex in subsets of the feasible region (note that the generalized convexity of these multiplicative functions has been studied in the literature for particular values of p and only for the strict quasiconvexity). In Section 3 we describe the “optimal level solutions” method and we state some preliminary results needed for the solution algorithm. In Section 4 the study of the problem is deepened on and several stopping criteria are stated with the aim to improve the solution algorithm. In Section 5 we finally propose a solution algorithm that solves the considered problems even when the feasible region is unbounded.

2. Definitions and Preliminary Results

In this paper we consider the following class of nonlinear multiplicative problems:

$$P : \begin{cases} \inf f(x) = \left(\frac{1}{2}x^T Qx + q^T x + q_0\right)\left(d^T x + d_0\right)^p \\ x \in X = \{x \in \mathbb{R}^n : Ax \geqslant b\} \end{cases}$$

where $A \in \mathbb{R}^{m\times n}$, $q, d \in \mathbb{R}^n$, $b \in \mathbb{R}^m$, $p, q_0, d_0 \in \mathbb{R}$, $p \neq 0$, $Q \in \mathbb{R}^{n\times n}$ is symmetric and positive definite and $d^T x + d_0 > 0\ \forall x \in X$.

For the sake of convenience, we define also the following subsets of the feasible region X:

$$\begin{aligned} X_{pos} &= \{x \in X : f(x) \geqslant 0\} = \left\{x \in X : \frac{1}{2}x^T Qx + q^T x + q_0 \geqslant 0\right\} \\ X_{neg} &= \{x \in X : f(x) \leqslant 0\} = \left\{x \in X : \frac{1}{2}x^T Qx + q^T x + q_0 \leqslant 0\right\} \end{aligned}$$

and the following value called the minimum feasible level:

$$\xi_{min} = \min_{x\in X}\left\{d^T x + d_0\right\}$$

Note that X_{neg} is a compact set and that $\xi_{min} > 0$ exists since X is closed and the linear function $d^T x + d_0$ is positive for all $x \in X$. Since $d^T x + d_0 > 0\ \forall x \in X$ problem P can be classified as follows:

– for $p < 0$ or $p \geqslant 1$ it is a convex multiplicative one,
– for $0 < p \leqslant 1$ it is a convex-concave multiplicative one.

In particular, if $p < 0$ then f is a fractional function, hence:

– for $-1 \leqslant p < 0$ P is a convex-concave fractional problem,
– for $p \leqslant -1$ problem P is a convex fractional one.

The generalized convexity of the objective function is an important property for problem P, both from the theoretical and the algorithmic point of view. This property implies that all the local minimum points are also global ones, and hence can be used as an efficient stopping criterion in solution algorithms.

Problem P is not a quasiconvex program in general; nevertheless it is possible to state some generalized convexity property of the objective function on the subsets X_{pos} and X_{neg} of the feasible region.

THEOREM 2.1. *Consider problem P. The following properties hold:*
(i) *if $p = -1$ then f is strictly pseudoconvex in X;*
(ii) *if $p < -1$ or $p > 0$ then f is strictly pseudoconvex in every convex subset of X_{neg};*
(iii) *if $-1 < p < 0$ then f is strictly pseudoconvex in every convex subset of X_{pos}.*

Proof. Let us denote the quadratic factor of f with $h(x) = (\frac{1}{2}x^T Qx + q^T x + q_0)$ and let us define on the set X the function $g(x) = (d^T x + d_0)^{-p}$, so that $f(x) = h(x)[g(x)]^{-1}$. First notice that g is convex for $p \leqslant -1$ or $p > 0$ and it is concave for $-1 \leqslant p < 0$ since its hessian matrix is

$$H_g(x) = \frac{p(p+1)}{(d^T x + d_0)^{p+2}} dd^T$$

Function f is strictly pseudoconvex in X if $\forall x, y \in X, x \neq y$, it is:

$$f(y) \leqslant f(x) \quad \Rightarrow \quad \nabla f(x)^T (y - x) < 0$$

Assume $f(y) \leqslant f(x)$; this condition can be rewritten as

$$h(y)[g(y)]^{-1} \leqslant h(x)[g(x)]^{-1}$$

that is, since $g(x) > 0 \ \forall x \in X$:

$$h(y) \leqslant h(x) \frac{g(y)}{g(x)}$$

Since Q is positive definite $h(x)$ is a strictly convex function, hence:

$$\nabla h(x)^T (y - x) < h(y) - h(x) \leqslant h(x) \left(\frac{g(y)}{g(x)} - 1 \right) = f(x)[g(y) - g(x)]$$

It results also:

$$\nabla f(x) = [g(x)]^{-1}\nabla h(x) - h(x)[g(x)]^{-2}\nabla g(x)$$

so that, since $g(x) > 0\ \forall x \in X$, we obtain:

$$\begin{aligned}\nabla f(x)^T(y-x) &= [g(x)]^{-1}\nabla h(x)^T(y-x) - h(x)[g(x)]^{-2}\nabla g(x)^T(y-x)\\ &< [g(x)]^{-1} f(x)[g(y)-g(x)] - f(x)[g(x)]^{-1}\nabla g(x)^T(y-x)\\ &= f(x)[g(x)]^{-1}[g(y)-g(x)-\nabla g(x)^T(y-x)]\end{aligned}$$

If $p = -1$ function g is affine, hence $g(y) - g(x) - \nabla g(x)^T(y-x) = 0$ and $\nabla f(x)^T(y-x) < 0$, so that f is strictly pseudoconvex in X.

If $p < -1$ or $p > 0$ function g is convex, hence $g(y)-g(x)-\nabla g(x)^T(y-x) \geqslant 0$; consequently for $f(x) \leqslant 0$ it is $\nabla f(x)^T(y-x) < 0$, that is to say that function f is strictly pseudoconvex in X_{neg}.

If $-1 < p < 0$ function g is concave, hence $g(y)-g(x)-\nabla g(x)^T(y-x) \leqslant 0$ and for $f(x) \geqslant 0$ it results $\nabla f(x)^T(y-x) < 0$, in other words function f is strictly pseudoconvex over X_{pos}. □

Note that Theorem 2.1 extends the results given in [4, 7, 20] which prove only the strict quasiconvexity of the function for the particular cases $p = -1$ and $p = -2$.

3. Optimal Level Solutions Approach

In this section we show how problem P can be solved by means of the optimal level solution approach [6, 8, 10, 12, 20], and we state some local optimality conditions useful to determine the solution algorithm.

Let $\xi \in \mathbb{R}$ be a real parameter, if we add to problem P the constraint $d^T x = \xi$ we have the following parametric strictly convex quadratic problem:

$$P_\xi : \begin{cases}\inf \left(\frac{1}{2}x^T Q x + q^T x + q_0\right)\xi^p\\ x \in X_\xi = \{x \in \mathbb{R}^n : Ax \geqslant b, d^T x + d_0 = \xi\}\end{cases}$$

The parameter ξ is said to be a *feasible level* if the set X_ξ is nonempty. An optimal solution of problem P_ξ is called an *optimal level solution*. For any given $\xi \in \mathbb{R}$, the optimal solution for problem P_ξ can be computed by means of any solution algorithm for strictly convex quadratic problems.

For the sake of completeness, let us now briefly recall the optimal level solution approach (see for example [12]). It is trivial that the optimal solution of problem P is also an optimal level solution and that, in particular, it is the optimal level solution with the smallest value; the idea of this approach is then to scan all the feasible levels, studying the corresponding optimal level solutions, until the minimizer of the problem is reached or a feasible halfline carrying $f(x)$ down to its infimum value is found.

Starting from an incumbent optimal level solution, this can be done by means of a sensitivity analysis on the parameter ξ, which allows us to move in the various steps through several optimal level solutions until the optimal solution is found.

Let us now study some optimality conditions useful to detect whether or not an optimal level solution is a local minimizer of problem P.

Let x' be an optimal solution of problem $P_{\xi'}$ and let $Nx = k$ be the equations of the constraints binding at x'. We can always select a subset of these constraints, making a submatrix M of N and correspondingly a subvector h of k, such that the rows of M and the vector d are linearly independent. As $P_{\xi'}$ is convex, x' is an optimal solution of $P_{\xi'}$ if and only if the following Kuhn-Tucker conditions are verified:

$$\begin{cases} Qx & -M^T\mu & -d\lambda & = -q \\ Mx & & & = h \\ d^T x & & & = \xi' - d_0 \end{cases} \tag{1}$$

where μ is the vector of the Lagrange multipliers associated to the constraints $Mx = h$ and λ is the Lagrange multiplier of the parametric constraint $d^T x = \xi' - d_0$.

The previous conditions can be rewritten in the following matrix form:

$$\begin{bmatrix} Q & -M^T & -d \\ M & 0 & 0 \\ d^T & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ \mu \\ \lambda \end{bmatrix} = \begin{bmatrix} -q \\ h \\ \xi' - d_0 \end{bmatrix} \tag{2}$$

where the matrix

$$S = \begin{bmatrix} Q & -M^T & -d \\ M & 0 & 0 \\ d^T & 0 & 0 \end{bmatrix}$$

is nonsingular since Q is positive definite and the rows of M and d are linearly independent. As a consequence, the solution of (1) is unique and is given by:

$$\begin{bmatrix} x' \\ \mu' \\ \lambda' \end{bmatrix} = S^{-1} \begin{bmatrix} -q \\ h \\ \xi' - d_0 \end{bmatrix}$$

Note also that, since x' is an optimal solution, we have $\mu' \geqq 0$.

Let us now carry on a sensitivity analysis on ξ' by means of a shifting value $\theta \in \mathbb{R}$; the unique solution of the Kuhn-Tucker system:

$$\begin{cases} Qx & -M^T\mu & -d\lambda & = -q \\ Mx & & & = h \\ d^T x & & & = \xi' + \theta - d_0 \end{cases}$$

is given by:

$$\begin{bmatrix} x'(\theta) \\ \mu'(\theta) \\ \lambda'(\theta) \end{bmatrix} = S^{-1} \begin{bmatrix} -q \\ h \\ \xi' + \theta - d_0 \end{bmatrix} = \begin{bmatrix} x' \\ \mu' \\ \lambda' \end{bmatrix} + \theta \begin{bmatrix} \Delta_x \\ \Delta_\mu \\ \Delta_\lambda \end{bmatrix}$$

where $[\Delta_x, \Delta_\mu, \Delta_\lambda]^T = S^{-1}[0, 0, 1]^T$; in other words $(\Delta_x, \Delta_\mu, \Delta_\lambda)$ is the unique solution of the linear system

$$\begin{cases} Qx & -M^T\mu & -d\lambda & = 0 \\ Mx & & & = 0 \\ d^T x & & & = 1 \end{cases}$$

so that $Q\Delta_x = M^T\Delta_\mu + d\Delta_\lambda$, $M\Delta_x = 0$, $d^T\Delta_x = 1$ and $\Delta_\lambda = \Delta_x^T Q\Delta_x$. Note also that, since $d^T\Delta_x = 1$ and Q is positive definite, it is:

$\Delta_x \neq 0$ and $\Delta_\lambda > 0$.

Let us now introduce the following sets:

- $F_R = \{\theta : x'(\theta) \in X\}$ *(feasibility range)*
- $O_R = \{\theta : \mu'(\theta) \geqq 0\}$ *(optimality range)*

Clearly, $x'(\theta)$ is an optimal level solution for $\theta \in F_R \cap O_R$.

Set $z' = \frac{1}{2}x'^T Qx' + q^T x' + q_0$ and

$$\begin{aligned} z(\theta) &= \left(\frac{1}{2}x'(\theta)^T Qx'(\theta) + q^T x'(\theta) + q_0\right)\left(\xi' + \theta\right)^p \\ &= \left(\frac{1}{2}\Delta_\lambda\theta^2 + \lambda'\theta + z'\right)\left(\xi' + \theta\right)^p . \end{aligned}$$

The first derivative of $z(\theta)$ is

$$\frac{dz}{d\theta}(\theta) = \left(\xi' + \theta\right)^{p-1}\left[\frac{1}{2}\Delta_\lambda(p+2)\theta^2 + ((p+1)\lambda' + \Delta_\lambda\xi')\theta + (pz' + \lambda'\xi')\right]$$

and hence:

- *If* $pz' + \lambda'\xi' > 0$ *[*$pz' + \lambda'\xi' < 0$*] then* $z(\theta)$ *is locally increasing [decreasing] at* $\theta = 0$.

Level optimality can be helpful also in studying local optimality, since a minimum point in a segment of optimal level solutions is a local minimizer of the problem. This fundamental property allows to prove the following conditions.

THEOREM 3.1. *Let* x' *be an optimal solution of problem* $P_{\xi'}$. *The following properties hold:*

(i) *if* $pz' + \lambda'\xi' = 0$ *and* $(p+1)\lambda' + \Delta_\lambda \xi' > 0$ *then* x' *is a local minimizer for problem* P*,*

(ii) *if* $p = -2$, $\Delta_\lambda \xi' > \lambda'$ *and* $\theta' = \frac{2z' - \lambda'\xi'}{\Delta_\lambda \xi' - \lambda'} \in F_R \cap O_R$ *then* $x'(\theta') = x' + \theta' \Delta_x$ *is a local minimizer for problem* P.

Suppose now that $p \neq -2$ *and that the equation* $\frac{dz}{d\theta}(\theta) = 0$ *has two different roots* $\theta_1 < \theta_2$*. The following further properties hold:*

(iii) *if* $p > -2$ *and* $\theta_2 \in F_R \cap O_R$ *then* $x'(\theta_2) = x' + \theta_2 \Delta_x$ *is a local minimizer for problem* P*,*

(iv) *if* $p < -2$ *and* $\theta_1 \in F_R \cap O_R$ *then* $x'(\theta_1) = x' + \theta_1 \Delta_x$ *is a local minimizer for problem* P*,*

Proof. The result follows just analyzing the local positivity and negativity of the derivative $\frac{dz}{d\theta}(\theta)$. □

Let us now focus on conditions regarding to vertices of the feasible polyhedron. We already pointed out, introducing the Kuhn-Tucker conditions (1), that for any vertex $x' \in X$ at least n constraints of X are binding as well as the parametric constraint $d^T x = \xi'$ and thus x' is a degenerate basic solution. For this reason $\mu'(\theta)$, $\lambda'(\theta)$, $z(\theta)$, F_R and O_R, actually depends on the chosen basis. Note that, since all the feasible levels must be examined in the algorithm, the used basis must contain the parametric constraint $d^T x = \xi'$.

To point out the described behavior, in the next results we will refer to the selected basis β using the notations $\mu'_\beta(\theta)$, $\lambda'_\beta(\theta)$ and so on. The next theorem follows directly from the formulae of $z(\theta)$ and its first derivative.

THEOREM 3.2. *Let* x' *be a vertex of* X *and suppose it is an optimal level solutions. If one of the following properties holds:*

(i) *there are two different bases* β_1 *and* β_2 *such that* $pz' + \lambda'_{\beta_1}\xi' > 0$, $\sup\{F_{R_{\beta_1}} \cap O_{R_{\beta_1}}\} > 0$, $pz' + \lambda'_{\beta_2}\xi' < 0$ *and* $\inf\{F_{R_{\beta_2}} \cap O_{R_{\beta_2}}\} < 0$,

(ii) $\sup\{F_{R_\beta} \cap O_{R_\beta}\} = 0$ *for any basis* β *such that* $pz' + \lambda'_\beta \xi' < 0$ *and* $\inf\{F_{R_\beta} \cap O_{R_\beta}\} = 0$ *for any basis* β *such that* $pz' + \lambda'_\beta \xi' > 0$,

then x' *is a local minimum for problem* P.

4. Stopping Criteria

Since X_{neg} is a compact set and f is continuous, it follows that $X_{neg} \neq \emptyset$ implies $\arg\min_{x \in X} f(x) = \arg\min_{x \in X_{neg}} f(x) \neq \emptyset$. For this reason, it is useful to determine conditions regarding the presence or absence of feasible points with nonpositive image.

DEFINITION 4.1. Consider problem P and its quadratic factor:

$$\left(\frac{1}{2}x^T Q x + q^T x + q_0\right)$$

From now on we denote with:

(i) $x_u = -Q^{-1}q$ (unconstrained minimum of the quadratic factor)

(ii) $\xi_u = d_0 - d^T Q^{-1} q = d^T x_u + d_0$ (level corresponding to x_u)

(iii) $q_u = q_0 - \frac{1}{2} q^T Q^{-1} q = \frac{1}{2} x_u^T Q x_u + q^T x_u + q_0$ (value of the quadratic factor at x_u)

(iv) $\delta = 2d^T Q^{-1} d$ ($\delta > 0$ since Q positive definite implies Q^{-1} positive definite)

LEMMA 4.1. *The following quadratic problem*

$$\begin{cases} \min \ \frac{1}{2} x^T Q x + q^T x + q_0 \\ d^T x + d_0 = \xi \end{cases}$$

attains the unconstrained minimum at

$$x(\xi) = -Q^{-1} \left(q - 2 \frac{\xi - \xi_u}{\delta} d \right)$$

with minimum value $\overline{Q}(\xi) = q_u + \frac{(\xi - \xi_u)^2}{\delta}$.

Proof. The minimum point of the problem verifies the following necessary and sufficient optimality condition:

$$\begin{cases} Qx + q = \lambda d \\ d^T x + d_0 = \xi \end{cases}$$

Since Q is positive definite it is also non singular, hence $x(\xi) = -Q^{-1}(q - \lambda d)$. By means of simple calculations, we then have:

$$\begin{aligned} \lambda &= 2 \frac{\xi - \xi_u}{\delta} \\ x(\xi) &= -Q^{-1} \left(q - 2 \frac{\xi - \xi_u}{\delta} d \right) \\ \overline{Q}(\xi) &= \frac{1}{2} x(\xi)^T Q x(\xi) + q^T x(\xi) + q_0 = \\ &= \frac{1}{2} \lambda^2 d^T Q^{-1} d + q_u = q_u + \frac{(\xi - \xi_u)^2}{\delta}. \end{aligned}$$ □

By means of Lemma 4.1 it possible to state the following conditions related to the positivity of the objective function f.

THEOREM 4.1. *Consider problem P. The following properties hold:*

(i) *if* $q_u > 0$ *then* $f(x) > 0 \ \forall x \in X$,

(ii) *if* $q_u \leqslant 0$ *and* $\xi_u + \sqrt{-\delta q_u} > 0$ *then*

$$f(x) \leqslant 0 \quad \Rightarrow \quad \xi_u - \sqrt{-\delta q_u} \leqslant d^T x + d_0 \leqslant \xi_u + \sqrt{-\delta q_u},$$

(iii) *if* $q_u \leqslant 0$ *and* $\xi_u + \sqrt{-\delta q_u} < \xi_{min}$ *then* $f(x) > 0\ \forall x \in X$.

Proof. i) The result follows trivially since $d^T x + d_0 > 0\ \forall x \in X$ and q_u is the unconstrained minimum value of the quadratic factor.

ii) Let $x \in X$ and $\xi > 0$ such that $d^T x + d_0 = \xi$. Condition $f(x) \leqslant 0$ implies $\overline{Q}(\xi) \leqslant 0$ and hence $(\xi - \xi_u)^2 \leqslant -\delta q_u$, that is to say that $\xi_u - \sqrt{-\delta q_u} \leqslant \xi \leqslant \xi_u + \sqrt{-\delta q_u}$.

iii) Follows directly from *ii)* since $d^T x + d_0 > 0\ \forall x \in X$. □

REMARK 4.1. Theorem 4.1 suggests a smart procedure to study problem P in the case

$$q_u \leqslant 0 \ \text{ and } \ \xi_u + \sqrt{-\delta q_u} \geqslant \xi_{min}$$

Split the feasible region X in the following subsets:

$$X_1 = X \cap \left\{x \in \mathbb{R}^n : \xi_u - \sqrt{-\delta q_u} \leqslant d^T x + d_0 \leqslant \xi_u + \sqrt{-\delta q_u}\right\}$$
$$X_2 = X \cap \left\{x \in \mathbb{R}^n : \xi_u - \sqrt{-\delta q_u} > d^T x + d_0\right\}$$
$$X_3 = X \cap \left\{x \in \mathbb{R}^n : d^T x + d_0 > \xi_u + \sqrt{-\delta q_u}\right\}$$

so that $X = X_1 \cup X_2 \cup X_3$. First solve the problem $\inf_{x \in X_1} f(x)$; if the minimum value computed is nonpositive, then it is also the minimum value of problem P, otherwise solve the two other problems $\inf_{x \in X_2} f(x)$ and $\inf_{x \in X_3} f(x)$ and compare the obtained results, taking into account that for $-1 \leqslant p \leqslant 0$ function f is strictly pseudo-convex on X_2 and X_3 and hence every local minimizer is a global one.

We conclude this section studying conditions which could be used as stopping criteria in algorithms solving problem P. With this aim, let us consider the following program associated to P:

$$\begin{cases} \min \ f(x) \\ d^T x + d_0 = \xi > 0 \end{cases}$$

The minimum point is attained again at $x(\xi)$ and the minimum values, associated to the levels $\xi > 0$, are given by:

$$\phi(\xi) = \frac{\xi^p}{\delta}\left[(\xi - \xi_u)^2 + \delta q_u\right] = \frac{\xi^{p+2}}{\delta}\left[\left(1 - \frac{\xi_u}{\xi}\right)^2 + \frac{\delta q_u}{\xi^2}\right]$$

By means of simple calculations, we obtain the corresponding first derivative:

$$\phi'(\xi) = \frac{\xi^{p-1}}{\delta}\left[(p+2)\xi^2 - 2\xi_u(p+1)\xi + p(\xi_u^2 + \delta q_u)\right]$$

which allow us to study the behavior of the unconstrained minimum level values. Note also that it results:

$$\lim_{\xi\to+\infty}\phi(\xi)=\begin{cases}+\infty & \text{if}\quad p>-2\\ \frac{1}{\delta} & \text{if}\quad p=-2\\ 0 & \text{if}\quad p<-2\end{cases}$$

Some optimality conditions, which could be used as stopping criteria in solution algorithms, can be provided when $\phi(\xi)$ is definitely increasing.

THEOREM 4.2. *Consider problem P, let $\xi_{min}>0$ be the minimum feasible level and let $x^*\in X$ and $\xi^*\geqslant\xi_{min}$ be such that $f(x^*)\leqslant\phi(\xi^*)$. If one of the following conditions holds:*

(i) $p>-2$ *and* $\xi_u^2\leqslant p(p+2)\delta q_u$,

(ii) $p>-2$, $\xi_u^2>p(p+2)\delta q_u$ *and* $\xi^*\geqslant\xi_u+\frac{-\xi_u+\sqrt{\xi_u^2-p(p+2)\delta q_u}}{p+2}$

(iii) $p=-2$, $\xi_u>0$ *and* $\xi^*\geqslant\frac{\xi_u^2+\delta q_u}{\xi_u}$,

(iv) $p=-2$, $\xi_u=0$ *and* $q_u<0$,

then $f(x^*)\leqslant f(x)$ $\forall x\in X$ such that $d^Tx+d_0\geqslant\xi^*$.

Proof. We prove the result showing that these conditions imply the increaseness of $\phi(\xi)$, function of the unconstrained minimum values associated to a feasible level ξ, so that

$$f(x^*)\leqslant\phi(\xi^*)\leqslant\phi(\xi)\leqslant f(x)\quad\forall x\in X\ \text{ such that }\ d^Tx+d_0=\xi\geqslant\xi^*$$

(i), (ii) Let $p>-2$. Since $\xi>0$ the derivative $\phi'(\xi)$ is nonnegative when

$$(p+2)\xi^2-2\xi_u(p+1)\xi+p(\xi_u^2+\delta q_u)\geqslant 0$$

Solving the second order inequality we have that for

$$\frac{\Delta}{4}=\xi_u^2-p(p+2)\delta q_u\leqslant 0$$

function $\phi(\xi)$ is increasing $\forall\xi>0$, while for $\frac{\Delta}{4}>0$ it is definitely increasing for

$$\xi\geqslant\xi_u+\frac{-\xi_u+\sqrt{\xi_u^2-p(p+2)\delta q_u}}{p+2}.$$

(iii), (iv) Let $p=-2$. It results:

$$\phi(\xi)=\frac{1}{\delta}\left[\left(1-\frac{\xi_u}{\xi}\right)^2+\frac{\delta q_u}{\xi^2}\right]$$

$$\phi'(\xi)=2\frac{\xi^{-3}}{\delta}\left[\xi_u\xi-(\xi_u^2+\delta q_u)\right]$$

hence if $\xi_u > 0$ function $\phi(\xi)$ is increasing for

$$\xi \geqslant \frac{\xi_u^2 + \delta q_u}{\xi_u},$$

while if $\xi_u = 0$ it is $\phi'(\xi) > 0$ just when $q_u < 0$. □

The previous result allows us to prove that only if $p \leqslant -2$ problem P may have no minimum points.

COROLLARY 4.1. *Consider problem P. The following property holds:*

$$\big(p > -2 \text{ or } X_{neg} \neq \emptyset\big) \quad \Rightarrow \quad \arg\min_{x \in X} f(x) \neq \emptyset$$

Proof. If $X_{neg} \neq \emptyset$ the result follows since f is continuous and X_{neg} is a compact set. Assume now $p > -2$ and let $x^* = \arg\min\{P_{\xi_{min}}\} \in X$; by means of some computations we can determine a level ξ^* such that $\xi^* \geqslant \xi_{min}$, $\phi(\xi^*) \geqslant f(x^*) = \min\{P_{\xi_{min}}\}$ and $\xi^* \geqslant \xi_u + \frac{-\xi_u + \sqrt{|\xi_u^2 - p(p+2)\delta q_u|}}{p+2}$. For Theorem 4.2 problem P is equivalent to the following one:

$$\begin{cases} \inf f(x) = \left(\frac{1}{2}x^T Qx + q^T x + q_0\right)\left(d^T x + d_0\right)^p \\ x \in Y = X \cap \{x \in \mathbb{R}^n : \xi_{min} \leqslant d^T x + d_0 \leqslant \xi^*\} \end{cases} \tag{3}$$

Let $\{x_k\} \subset Y$ be a sequence such that $f(x_k) \to \inf_{x \in X} f(x)$ and define the corresponding sequence $\{y_k\} \subset [\xi_{min}, \xi^*]$ such that $y_k = d^T x_k + d_0$. Since $[\xi_{min}, \xi^*]$ is a compact set we can extract a subsequence $\{x_j\} \subset \{x_k\}$ such that

$$f(x_j) \to \inf_{x \in X} f(x)$$

and

$$d^T x_j + d_0 \to \overline{\xi} \in [\xi_{min}, \xi^*].$$

Since f is continuous, $\arg\min\{P_{\overline{\xi}}\} \neq \emptyset$ and $d^T x_j + d_0 \to \overline{\xi}$ it is:

$$\inf_{x \in X} f(x) \leqslant \min\{P_{\overline{\xi}}\} \leqslant \lim_{j \to +\infty} f(x_j) = \inf_{x \in X} f(x)$$

and hence $\arg\min_{x \in X} f(x) = \arg\min\{P_{\overline{\xi}}\}$.

5. A Solution Algorithm

In order to find the infimum/minimum of P it would be necessary to solve problem P_ξ for all the feasible levels. In this section we will show that this can be done by means of a finite number of iterations, using the results of the previous sections.

5.1. MAIN STEPS

The algorithm starts from the minimum level ξ_{min} and then scans all the greater ones looking for the optimal solution.

Algorithm Structure

(0) Compute, by means of two linear programs, the values:

$$\xi_{min} := \min_{x \in X} d^T x + d_0 \quad , \quad \xi_{max} := \sup_{x \in X} d^T x + d_0$$

and determine ξ_u, q_u and δ.

(1) Let $\xi' := \xi_{min}$ be the starting feasible level; $x' := \arg\min\{P_{\xi'}\}$; $UB := f(x')$; $x^* := x'$; unbounded:$= false$; stop:$= false$;

(2) While not stop do

(2a) With respect to ξ' and x' determine μ', λ', Δ_x, Δ_μ, Δ_λ, $\sup F_R$, $\sup O_R$; $\theta_m := \min\{\sup F_R, \sup O_R\}$;

(2b) Determine the next level $\tilde{\xi} > \xi'$, the best optimal level solution $\overline{x}$ for the levels $\xi \in [\xi', \tilde{\xi}]$, and test the unboundedness;

(2c) If unbounded$= true$ then stop:$= true$
else begin

- If $f(\overline{x}) < UB$ then $x^* := \overline{x}$ and $UB := f(\overline{x})$;
- If one of the following conditions holds:

  * $\tilde{\xi} \geqslant \xi_{max}$
  * $UB \leqslant 0$ and $\tilde{\xi} > \xi_u + \sqrt{-\delta q_u}$
  * $UB \leqslant \phi(\tilde{\xi})$ and $p > -2$ and

$$\text{not} \left\{ \xi_u^2 > p(p+2)\delta q_u \text{ and } \tilde{\xi} < \xi_u + \frac{-\xi_u + \sqrt{\xi_u^2 - p(p+2)\delta q_u}}{p+2} \right\}$$

  * $UB \leqslant \phi(\tilde{\xi})$ and $p = -2$ and

$$\left[\left(\xi_u > 0 \text{ and } \tilde{\xi} \geqslant \frac{\xi_u^2 + \delta q_u}{\xi_u} \right) \text{ or } (\xi_u = 0 \text{ and } q_u < 0) \right]$$

then stop:$= true$
else begin

- $\gamma := \tilde{\xi} - \xi'$; $\xi' := \tilde{\xi}$;
- if $\gamma > \sup F_R$ then $x' := \arg\min\{P_{\xi'}\}$ else $x' := x' + \gamma \Delta_x$;

end;

end;

(3) If unbounded= $true$ then $\inf_{x\in X} f(x) = Inf\_Val$ else x^* is the optimal solution for problem P. □

Note that in all the iterations the variable UB gives an upper bound for the optimal value with respect to the levels $\xi > \xi'$, while x^* is the best optimal level solution with respect to the levels $\xi \leqslant \xi'$. Note also that in *2c)* the various stopping criteria studied in the previous sections have been used.

5.2. MOVING STEPS

It remains to show how to implement step *2b)* in the previous procedure. With this aim, first note that for all $\theta \in O_R$, the value $z(\theta)$ is a lower bound for the parametric problem $P_{\xi'+\theta}$; in fact if $\theta \in F_R$ then $x'(\theta)$ is an optimal level solution, otherwise (if $\theta \notin F_R$) $x'(\theta)$ is unfeasible for $P_{\xi'+\theta}$ but is an optimal solution of a problem with the same objective function as $P_{\xi'+\theta}$ and a feasible region containing $X_{\xi'+\theta}$. Recall finally that the derivative of $z(\theta)$ is

$$\frac{dz}{d\theta}(\theta)=\left(\xi'+\theta\right)^{p-1}\left[\frac{1}{2}\Delta_\lambda(p+2)\theta^2+((p+1)\lambda'+\Delta_\lambda\xi')\theta+(pz'+\lambda'\xi')\right]$$

Case A: $\mathbf{p > -2}$

(A1) Compute $\Delta := ((p+1)\lambda' + \Delta_\lambda\xi')^2 - 2\Delta_\lambda(p+2)(pz' + \lambda'\xi')$;
 if $\Delta > 0$ then $\theta_2 := \frac{-((p+1)\lambda'+\Delta_\lambda\xi')+\sqrt{\Delta}}{\Delta_\lambda(p+2)}$ end if;

(A2) One of the following exhaustive cases occurs:

 (A2a) $[\Delta \leqslant 0$ or $(\Delta > 0$ and $\theta_2 \leqslant 0)]$, that is $z(\theta)$ is increasing for $\theta \geqslant 0$. Then $\overline{x} := x'$ and $\tilde{\xi} := \xi' + \sup O_R$;

 (A2b) $[\Delta > 0$ and $\theta_2 > 0]$, that is $z(\theta)$ is decreasing for $\theta \in [\theta_1, \theta_2]$ and increasing elsewhere.
 If $\theta_2 \leqslant \theta_m$
  then $\overline{x} := x' + \theta_2\Delta_x$ and $\tilde{\xi} := \xi' + \sup O_R$
  else if $\theta_2 \leqslant \sup O_R$ and $UB \leqslant z(\theta_2)$
   then $\overline{x} := x'$ and $\tilde{\xi} := \xi' + \sup O_R$;
   else $\overline{x} := x' + \theta_m\Delta_x$ and $\tilde{\xi} := \xi' + \theta_m$
  end if
 end if. □

Case B: $\mathbf{p < -2}$

(B1) Compute $\Delta := ((p+1)\lambda' + \Delta_\lambda\xi')^2 - 2\Delta_\lambda(p+2)(pz' + \lambda'\xi')$;
 if $\Delta > 0$
  then $\theta_1 := \frac{-((p+1)\lambda'+\Delta_\lambda\xi')-\sqrt{\Delta}}{\Delta_\lambda(p+2)}$;
   if $\theta_1 > 0$
    then $\overline{x} := x' + \min\{\theta_m, \theta_1\}\Delta_x$;
     if $f(\overline{x}) < UB$ then $x^* := \overline{x}$; $UB := f(\overline{x})$ end if
   end if
 end if;

(B2) One of the following exhaustive cases occurs:

(B2a) [sup $O_R = +\infty$ and $UB \leqslant 0$]. Then $\overline{x} := x'$ and $\tilde{\xi} := +\infty$;

(B2b) [sup $O_R = +\infty$ and $UB > 0$]
If sup $F_R = +\infty$
then unbounded:= $true$ and $Inf\_Val := 0$
else $\overline{x} := x' + (\sup F_R)\Delta_x$; $\tilde{\xi} := \xi' + \sup F_R$
end if;

(B2c) [sup $O_R < +\infty$]. Then $\overline{x} := x'$;
if $z(\sup O_R) \geqslant UB$
then $\tilde{\xi} := \xi' + \sup O_R$ else $\tilde{\xi} := \xi' + \sup F_R$
end if; □

Note, finally, that in the case $p = -2$ the derivative of $z(\theta)$ is

$$\frac{dz}{d\theta}(\theta) = \left(\xi' + \theta\right)^{-3}\left[(\Delta_\lambda \xi' - \lambda')\theta + (\lambda'\xi' - 2z')\right]$$

Case C: $\mathbf{p = -2}$

(C1) Compute $\alpha_1 := \Delta_\lambda \xi' - \lambda'$ and $\alpha_0 := \lambda'\xi' - 2z'$;

(C2) One of the following exhaustive cases occurs:

(C2a) [($\alpha_0 > 0$ and $\alpha_1 \geqslant 0$) or ($\alpha_0 = 0$ and $\alpha_1 > 0$)].
Then $\overline{x} := x'$ and $\tilde{\xi} := \xi' + \sup O_R$;

(C2b) [($\alpha_0 \geqslant 0$ and $\alpha_1 < 0$) or ($\alpha_0 < 0$ and $\alpha_1 \leqslant 0$)].
If $UB \leqslant \frac{1}{2}\Delta_\lambda$
then $\overline{x} := x'$ and $\tilde{\xi} := \xi' + \sup O_R$
else if $\theta_m = +\infty$
then unbounded:= $true$ and $Inf\_Val := \frac{1}{2}\Delta_\lambda$;
else $\overline{x} := x' + \theta_m \Delta_x$ and $\tilde{\xi} := \xi' + \theta_m$
end if
end if;

(C2c) [$\alpha_0 < 0$ and $\alpha_1 > 0$].
Compute $\hat{\theta} := \frac{-\alpha_0}{\alpha_1}$;
if sup $O_R < \hat{\theta}$
then $\overline{x} := x' + \theta_m \Delta_x$ and $\tilde{\xi} := \xi' + \theta_m$
else if $UB \leqslant z(\hat{\theta})$
then $\overline{x} := x'$ and $\tilde{\xi} := \xi' + \sup O_R$;
else if sup $F_R > \hat{\theta}$
then $\overline{x} := x' + \hat{\theta}\Delta_x$ and $\tilde{\xi} := \xi' + \sup O_R$
else $\overline{x} := x' + (\sup F_R)\Delta_x$ and $\tilde{\xi} := \xi' + \sup F_R$
end if
end if
end if. □

5.3. CORRECTNESS AND FINITENESS

The correctness of the proposed algorithm follows since all the feasible levels are scanned and the optimal solution, if it exists, is also an optimal level solution.

It remains to verify the convergence (finiteness), that is to say that the procedure stops after a finite number of steps. First note that, at every iterative step of the proposed algorithm, the set of binding constraints changes; note also that the level is increased from ξ' to $\tilde{\xi}$ so that it is not possible to obtain again an already used set of binding constraints; the convergence then follows since:

- we have a finite number of possible sets of binding constraints,
- the algorithm detects the infimum value of f recognizing a feasible extreme ray of optimal level solutions.

In particular, let us note that at every iterative step at least one used constraint is deleted and some new constraints may be added:

- if $\tilde{\xi} := \xi' + \sup F_R$, $\sup F_R \leqslant \sup O_R$, then one constraint is deleted and one new constraint is added,
- if $\tilde{\xi} := \xi' + \sup O_R$, $\sup O_R < \sup F_R$, then at least one constraint is deleted and no new constraint is added,
- if $\tilde{\xi} := \xi' + \sup O_R$, $\sup F_R < \sup O_R$, then at least one constraint is deleted and some new constraints may be added.

REMARK 5.1. It is worth comparing the algorithm proposed in this paper with the one (which we will refer to as KK-algorithm) studied in [16] and related to convex multiplicative programs.

In the particular case $p \notin]0,1[$, X bounded and $f(x) \geqslant 0 \ \forall x \in X$, problem P is a special case of the problems studied in [16] and hence can be solved with KK-algorithm.

Note that KK-algorithm is based on the solution of the following master problem:

$$K_\xi : \begin{cases} \min \xi \left(\dfrac{1}{2}x^T Qx + q^T x + q_0\right) + \frac{1}{\xi}\left(d^T x + d_0\right)^p \\ x \in X,\ \xi > 0 \end{cases}$$

by means of a branch-and-bound approach. As a consequence, the algorithm proposed in this paper is different since:

- it is based on simplex-like steps, while KK-algorithm uses a branch-and-bound scheme,
- P_ξ is a strictly convex quadratic problem $\forall p \in \mathbb{R}$, while K_ξ is convex but not necessarily quadratic (it is quadratic just for $p = 0, 1, 2$),

Recall finally that if $0 < p \leqslant 1$ or $X_{neg} \neq \emptyset$ or X is unbounded, KK-algorithm cannot be used to solve problem P.

5.4. EXAMPLES

In this section we show how the proposed algorithm can be used to solve the following problems:

$$\begin{cases} \inf\ f(x) = \left(\frac{1}{2}[x_1, x_2]\begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + [-2, 1]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - 4\right)(x_1 + 2x_2 + 1)^p \\ {\scriptstyle (1)}\ x_1 \geqslant 0\ ,\ {\scriptstyle (2)}\ x_2 \geqslant 0\ ,\ {\scriptstyle (3)}\ x_1 - x_2 \geqslant -2\ ,\ {\scriptstyle (4)}\ -x_1 + 2x_2 \geqslant -4 \end{cases}$$

where $p = 3, -3, -2$ (for the sake of simplicity we will not use the stopping criteria in step *2c)*). The parametric constraint results to be:

$$ {\scriptstyle (p)}\ x_1 + 2x_2 + 1 = \xi'\ ,\ \xi' \in \mathbb{R}.$$

We first have to consider the following preliminary linear programs:

$$\begin{cases} \inf / \sup\ x_1 + 2x_2 + 1 \\ x_1 \geqslant 0\ ,\ x_2 \geqslant 0\ ,\ x_1 - x_2 \geqslant -2\ ,\ -x_1 + 2x_2 \geqslant -4 \end{cases}$$

the optimal solution of the minimum problem is $(0, 0)$, hence $\xi_{min} = 1$; the maximum problem is unbounded, hence $\xi_{max} = +\infty$.

EXAMPLE 5.1. (Case $p = 3$). *The following steps are obtained:*

(1) $\xi' := \xi_{min} = 1$*;* $x' := (0, 0)$*;* $UB := f(0, 0) = -4$*;* $x^* := (0, 0)$*; unbounded:=* $false$*; stop:=* $false$*;*

(2) $\beta = \{(2), (p)\}$*;* $\mu' := 5$*,* $\lambda' := -2$*,* $\Delta_x := (1, 0)$*,* $\Delta_\mu := -5$*,* $\Delta_\lambda := 3$*,* $\sup F_R := 4$*,* $\sup O_R := 1$*;* $\theta_m := 1$*;* $\Delta := 445$*;* $\theta_2 := 1.73966$*;*
- ∘ *in the moving step case A2b) occurs with* $\theta_2 > \sup O_R = \theta_m$*;*
- ∘ $\overline{x} := (0, 0) + 1(1, 0) = (1, 0)$*;* $\tilde{\xi} := 1 + 1 = 2$*;*
- ∘ $f(\overline{x}) = -36 < UB$ *hence* $x^* := (1, 0)$ *and* $UB := -36$*;*
- ∘ $\xi' := 2$*;* $\gamma = 1 < 4 = \sup F_R$ *hence* $x' := (0, 0) + 1(1, 0) = (1, 0)$*;*

(3) $\beta = \{(p)\}$*;* $\lambda' := 1$*,* $\Delta_x := (0, \frac{1}{2})$*,* $\Delta_\lambda := \frac{1}{2}$*,* $\sup F_R := 6$*,* $\sup O_R := +\infty$*;* $\theta_m := 6$*;* $\Delta := 82.5$*;* $\theta_2 := 1.63318$*;*
- ∘ *in the moving step case A2b) occurs with* $\theta_2 < \theta_m$*;*
- ∘ $\overline{x} := (1, 0) + \theta_2(0, \frac{1}{2}) = (1, 0.8166)$ *and* $\tilde{\xi} := \xi' + \sup O_R = +\infty$*;*
- ∘ $f(\overline{x}) = -105.5 < UB$ *hence* $x^* := (1, 0.8166)$ *and* $UB := -105.5$*;*
- ∘ $\tilde{\xi} \geqslant \xi_{max}$ *hence stop:=* $true$*;*

(4) $x^* := (1, 0.8166)$ *is the optimal solution with* $f(x^*) = -105.5$. □

EXAMPLE 5.2. (Case $p = -3$). *The following steps are obtained:*

(1) $\xi' := \xi_{min} = 1$*;* $x' := (0, 0)$*;* $UB := f(0, 0) = -4$*;* $x^* := (0, 0)$*; unbounded:=* $false$*; stop:=* $false$*;*

(2) $\beta = \{(2),(p)\}$; $\mu' := 5$, $\lambda' := -2$, $\Delta_x := (1,0)$, $\Delta_\mu := -5$, $\Delta_\lambda := 3$, $\sup F_R := 4$, $\sup O_R := 1$; $\theta_m := 1$; $\Delta := 109$; $\theta_1 := -1.146768$;
- ∘ *in the moving step case B2c) occurs;*
- ∘ $\overline{x} := x' = (0,0)$; $z(\sup O_R) = \frac{-9}{16} > UB = -4$ *hence* $\tilde{\xi} := \xi' + \sup O_R = 2$;
- ∘ $f(\overline{x}) \geqslant UB$ *hence* x^* *and* UB *are not updated;*
- ∘ $\xi' := 2$; $\gamma = 1 < 4 = \sup F_R$ *hence* $x' := (0,0) + 1(1,0) = (1,0)$;

(3) $\beta = \{(p)\}$; $\lambda' := 1$, $\Delta_x := (0, \frac{1}{2})$, $\Delta_\lambda := \frac{1}{2}$, $\sup F_R := 6$, $\sup O_R := +\infty$; $\theta_m := 6$; $\Delta := 16.5$; $\theta_1 := -10.124038$;
- ∘ *in the moving step case B2a) occurs;*
- ∘ $\overline{x} := x' = (1,0)$ *and* $\tilde{\xi} := +\infty$;
- ∘ $f(\overline{x}) \geqslant UB$ *hence* x^* *and* UB *are not updated;*
- ∘ $\tilde{\xi} \geqslant \xi_{max}$ *hence stop*:= $true$;

(4) $x^* := (0,0)$ *is the optimal solution with* $f(x^*) = -4$. □

EXAMPLE 5.3. (Case $p = -2$). *The following steps are obtained:*

(1) $\xi' := \xi_{min} = 1$; $x' := (0,0)$; $UB := f(0,0) = -4$; $x^* := (0,0)$; *unbounded*:= $false$; *stop*:= $false$;

(2) $\beta = \{(2),(p)\}$; $\mu' := 5$, $\lambda' := -2$, $\Delta_x := (1,0)$, $\Delta_\mu := -5$, $\Delta_\lambda := 3$, $\sup F_R := 4$, $\sup O_R := 1$; $\theta_m := 1$; $z' := -4$; $\alpha_0 := 6$; $\alpha_1 := 5$;
- ∘ *in the moving step case C2a) occurs;*
- ∘ $\overline{x} := x' = (0,0)$; $\tilde{\xi} := \xi' + \sup O_R = 2$;
- ∘ $f(\overline{x}) \geqslant UB$ *hence* x^* *and* UB *are not updated;*
- ∘ $\xi' := 2$; $\gamma = 1 < 4 = \sup F_R$ *hence* $x' := (0,0) + 1(1,0) = (1,0)$;

(3) $\beta = \{(p)\}$; $\lambda' := 1$, $\Delta_x := (0, \frac{1}{2})$, $\Delta_\lambda := \frac{1}{2}$, $\sup F_R := 6$, $\sup O_R := +\infty$; $\theta_m := 6$; $z' := \frac{-9}{2}$; $\alpha_0 := 11$; $\alpha_1 := 0$;
- ∘ *in the moving step case C2a) occurs;*
- ∘ $\overline{x} := x' = (1,0)$ *and* $\tilde{\xi} := \xi' + \sup O_R = +\infty$;
- ∘ $f(\overline{x}) \geqslant UB$ *hence* x^* *and* UB *are not updated;*
- ∘ $\tilde{\xi} \geqslant \xi_{max}$ *hence stop*:= $true$;

(4) $x^* := (0,0)$ *is the optimal solution with* $f(x^*) = -4$. □

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